ON THE SPECTRUM OF BARGMANN-TOEPLITZ OPERATORS WITH SYMBOLS OF A VARIABLE SIGN

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ABSTRACT. The paper discusses the spectrum of Toeplitz operators in Bargmann spaces. Our Toeplitz operators have real symbols with a variable sign and a compact support. A class of examples is considered where the asymptotics of the eigenvalues of such operators can be computed. These examples show that this asymptotics depends on the geometry of the supports of the positive and negative parts of the symbol. Applications to the perturbed Landau Hamiltonian are given.

1. INTRODUCTION

1.1. **Bargmann-Toeplitz operators.** Let $\mathcal{F}^2$ be the Fock space, i.e. the Hilbert space of all entire functions $f = f(z)$, $z \in \mathbb{C}$ such that

$$\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dm(z) \equiv \|f\|_{\mathcal{F}^2}^2 < \infty.$$

Here and in what follows $dm(z)$ denotes the Lebesgue measure in $\mathbb{C}$. Let $P_{\mathcal{F}^2}$ be the orthogonal projection in $L^2(\mathbb{C}, e^{-|z|^2} dm(z))$ onto $\mathcal{F}^2$. For a bounded compactly supported function $V : \mathbb{C} \to \mathbb{R}$, the *Bargmann-Toeplitz operator T_V in $\mathcal{F}^2$* is the operator

$$T_V : \mathcal{F}^2 \to \mathcal{F}^2, \quad T_V f = P_{\mathcal{F}^2} V f, \quad f \in \mathcal{F}^2. \tag{1.1}$$

The function V is called the *symbol* of the operator T_V. The quadratic form of the operator T_V is given by

$$(T_V f, f)_{\mathcal{F}^2} = \int_{\mathbb{C}} V(z) |f(z)|^2 e^{-|z|^2} dm(z), \quad f \in \mathcal{F}^2. \tag{1.2}$$

It is well known that the operator T_V is compact, see e.g. [11, 17]. The subject of this paper is the rate of accumulation of the eigenvalues of T_V to zero for compactly supported symbols V of a variable sign.

For symbols V with a moderate decay at infinity, the asymptotics of the eigenvalues of T_V has been extensively studied in the literature, see [15, 16, 21, 22] and also [8, Chaps. 11 and 12]. For this class of symbols, there appears to be very little difference between the case of non-negative V and the case of V of a variable sign; this concerns both the results and the technique required for their proof. As it will be clear from the discussion below, the situation for compactly supported symbols V is radically different.

We denote by $\lambda_1^+ \geq \lambda_2^+ \geq \cdots > 0$ the sequence of positive eigenvalues of T_V enumerated with multiplicities taken into account. If T_V has only k positive eigenvalues, we set $\lambda_n^+ = 0$ for $n \geq k + 1$. Similarly, let $\lambda_1^- \geq \lambda_2^- \geq \cdots > 0$ be the sequence of positive eigenvalues of $-T_V = T_{-V}$, appended by zeros if necessary. Finally, $s_1 \geq s_2 \geq \cdots > 0$ is the sequence

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of the singular values of T_V; of course, this is just the re-ordered union of the non-zero elements of the sequences $\{\lambda_n^+\}_{n=1}^\infty$ and $\{\lambda_n^-\}_{n=1}^\infty$.

1.2. **The case** $V \geq 0$. In this case we have $T_V \geq 0$ and so $\lambda_n^- \equiv 0$ and $\lambda_n^+ = s_n$. Let us first briefly discuss the results known for the case of V with a moderate decay at infinity. Suppose that

$$V(x) = \mathbf{v}(x/|x|)|x|^{-\rho}(1 + o(1)) \quad \text{as } |x| \to \infty, \tag{1.3}$$

where $\rho > 0$ is a constant and $\mathbf{v} \geq 0$ is a function on the unit circle which satisfies some regularity assumptions. Then one has [21, 15] $s_n = Cn^{-\rho/2}(1 + o(1))$, as $n \to \infty$, where the constant C can be explicitly expressed in terms of $\mathbf{v}$ and ρ.

Now suppose that V is compactly supported. Then the above result, in combination with variational considerations, says only that s_n decays faster than any power of n as $n \to \infty$. In fact, it turns out that the rate of decay of s_n as $n \to \infty$ is super-exponential. In [17, 13] it was proved that, if compactly supported $V \geq 0$ is separated from zero on some open set (which is true, for example, if V is continuous and not identically zero), then

$$\log s_n = -n \log n + O(n), \quad n \to \infty, \tag{1.4}$$

or, using a somewhat informal but more expressive notation,

$$s_n = e^{-n \log n + O(n)}, \quad n \to \infty. \tag{1.5}$$

More precise asymptotics of s_n is known [4]; we will say more about this in Section 2.2.

For future reference we would like to display a corollary of (1.4), (1.5) which concerns the eigenvalue counting function. For a self-adjoint operator A and an interval $\Delta \subset \mathbb{R}$, we denote by $E(\Delta; A)$ the spectral projection of A corresponding to Δ and by $N(\Delta; A) = \operatorname{rank} E(\Delta; A)$ the total number of eigenvalues (counting multiplicities) of A in Δ. With this notation, (1.5) yields the asymptotic relation

$$N((\lambda, \infty); T_V) = \frac{|\log \lambda|}{\log|\log \lambda|}(1 + o(1)), \quad \lambda \to +0, \quad V \geq 0. \tag{1.6}$$

Note that (1.6) follows from (1.5) but the converse statement is false.

1.3. **The case of** V **of a variable sign.** The main focus of this paper is the case of symbols V of a variable sign. We denote $V_\pm = (|V| \pm V)/2 \geq 0$. If V has a power asymptotics at infinity (1.3) with $\mathbf{v}$ of a variable sign, then one has [21, 15] $\lambda_n^\pm = C^\pm n^{-\rho/2}(1 + o(1))$, as $n \to \infty$, where the constants $C^\pm$ can be explicitly expressed in terms of $\mathbf{v}_\pm$ and ρ. Essentially, in this case the operators T_{V_+} and T_{V_-} are "asymptotically orthogonal". This means that the asymptotics of λ_n^+ (resp. λ_n^-) is determined by the positive (resp. negative) part of the symbol V only. See also [2] for some older results of a similar nature.

Now suppose that the symbol V is compactly supported. As we shall see, T_{V_+} and T_{V_-} are no longer asymptotically orthogonal in this case and the question of the asymptotics of T_V becomes more complicated. Of course, by the estimates $-T_{V_-} \leq T_V \leq T_{V_+}$, the min-max principle shows that we still have the super-exponential upper bounds of the type (1.5) on both positive and negative eigenvalues of T_V:

$$\lambda_n^\pm \leq e^{-n \log n + O(n)}, \quad n \to \infty.$$

However, if both V_+ and V_- are non-zero, the lower bounds on the eigenvalues of T_V become a delicate problem because of the possible cancellations between the contributions of V_+

and V_-. Even the question of whether there are infinitely many non-zero eigenvalues of T_V in this case is non-trivial. It was only in 2008 that the following theorem has been established by D. Luecking [12]:

Theorem ([12]). *If T_V is an operator of a finite rank, then $V \equiv 0$.*

Thus, for $V \not\equiv 0$ there are always infinitely many non-zero eigenvalues. The purpose of this paper is to show by means of two simple "extreme" examples that the asymptotic behaviour of the eigenvalues $\lambda_n^\pm$ as $n \to \infty$ is determined by the geometry of the sets $\operatorname{supp} V_+$ and $\operatorname{supp} V_-$. In Section 2 we show that if $\operatorname{supp} V_-$ is "locked inside" $\operatorname{supp} V_+$, then the number of negative eigenvalues of T_V is finite. We prove a simple

Theorem 1.1. *Suppose that V is continuous and $K = \operatorname{supp}(V)$ is connected. Suppose also that there exists a smooth closed simple curve $\Gamma \subset K$ such that $V(x) \geq \delta > 0$ on Γ and $\operatorname{supp} V_-$ lies inside the curve Γ. Then V has only finitely many negative eigenvalues.*

Of course, by Luecking's theorem, the number of positive eigenvalues in this case is infinite. In Section 2 we prove Theorem 1.1 and also discuss sufficient conditions on V which ensure that T_V has infinitely many negative eigenvalues.

At the other extreme, in Section 3 we consider a simple class of symbols V with the sets $\operatorname{supp} V_+$ and $\operatorname{supp} V_-$ "well separated" and show that in this case there are infinitely many positive eigenvalues and infinitely many negative eigenvalues. We also establish some asymptotic bounds on the sequences $\lambda_n^\pm$ in this case, see next subsection.

Finally, in Section 4 we briefly discuss applications of the above results to the perturbed Landau Hamiltonian, i.e. to the two-dimensional Schrödinger operator with a constant homogeneous magnetic field. This was, in fact, the initial motivation for this work.

1.4. **Asymptotics bounds on the spectrum of T_V.** Here we discuss the results concerning the case of V with the supports of V_+ and V_- being "well separated". Let Ω_+ and Ω_- be two compact domains in $\mathbb{C}$ with Lipschitz boundaries. We denote by $\operatorname{Co}(\Omega_\pm)$ the closed convex hull of $\Omega_\pm$, and we assume that

$$\operatorname{Co}(\Omega_+) \cap \operatorname{Co}(\Omega_-) = \varnothing. \tag{1.7}$$

We consider the symbols V of the form

$$V(z) = v_+(z)\chi_{\Omega_+}(z) - v_-(z)\chi_{\Omega_-}(z), \tag{1.8}$$

where $v_\pm$ are bounded non-negative functions such that $\inf_{\Omega_\pm} v_\pm > 0$. In Section 3 we establish our main result:

Theorem 1.2. *Let Ω_+, Ω_- be Lipschitz domains in $\mathbb{C}$ which satisfy (1.7) and let V be of the form (1.8). Then for the singular values s_n of T_V the asymptotics (1.5) holds true. Moreover, there exist constants $0 < \delta_\pm \leq \Delta_\pm < 1$ with $\delta_+ + \Delta_- = 1$ and $\delta_- + \Delta_+ = 1$ such that the asymptotic estimates*

$$\exp\left(-\frac{n \log n}{\delta_\pm} + o(n \log n)\right) \leq \lambda_n^\pm \leq \exp\left(-\frac{n \log n}{\Delta_\pm} + o(n \log n)\right), \quad n \to \infty \tag{1.9}$$

hold true.

Theorem 1.2 yields the following estimates for the eigenvalue counting functions:

$$N((\lambda, \infty); |T_V|) \equiv N((\lambda, \infty); T_V) + N((\lambda, \infty); -T_V) = \frac{|\log \lambda|}{\log |\log \lambda|}(1 + o(1)), \quad \lambda \to +0,$$
(1.10)

$$\delta_\pm \frac{|\log \lambda|}{\log |\log \lambda|}(1 + o(1)) \leq N((\lambda, \infty); \pm T_V) \leq \Delta_\pm \frac{|\log \lambda|}{\log |\log \lambda|}(1 + o(1)), \quad \lambda \to +0. \quad (1.11)$$

It would be interesting to study the optimal constants $\delta_\pm$, $\Delta_\pm$ and analyse their dependence on the pair of domains Ω_+, Ω_-. Unfortunately, we do not have anything to say about this except in some very special case. Assume that Ω_+ and Ω_- can be interchanged by a Euclidean motion in $\mathbb{C}$. More precisely, suppose that for some $\theta \in \mathbb{R}$ and $\mathbf{c} \in \mathbb{C}$, we can set $\varphi(z) = e^{i\theta}z + \mathbf{c}$ or $\varphi(z) = e^{i\theta}\bar{z} + \mathbf{c}$ and get $\varphi(\Omega_+) = \Omega_-$ and $\varphi(\Omega_-) = \Omega_+$.

Theorem 1.3. *Assume the hypotheses of Theorem 1.2 and suppose that $\Omega_\pm$ possess the above described symmetry. Then $\delta_+ = \delta_- = \Delta_+ = \Delta_- = 1/2$ in* (1.9). *Moreover, the remainder estimate in* (1.9) *can be improved; one has*

$$\lambda_n^\pm = e^{-2n \log n + O(n)}, \quad n \to \infty. \tag{1.12}$$

In our construction, the constants $\delta_\pm$, $\Delta_\pm$ arise through the spectral analysis of some auxiliary operators related to the pair of domains Ω_+, Ω_-, see Section 3.1. By analogy with the embedding operators, we call them the "outbedding operators". The study of the spectral properties of these "outbedding operators" seems to be an interesting problem on its own.

1.5. **Notation.** For two domains $\Omega_\pm$ we denote by $\| \cdot \|_\pm$ and $(\cdot, \cdot)_\pm$ the norm and inner product in $L^2(\Omega_\pm, dm(z))$. For a domain $\Omega \subset \mathbb{C}$, the set Ω^c is the complement of Ω in $\mathbb{C}$. Notation $\sigma_{\text{ess}}(A)$ stands for the essential spectrum of self-adjoint operator A. For a closed simple Jordan curve $\Gamma \in \mathbb{C}$, we denote by $\text{Int}\,\Gamma$ (resp. $\text{Ext}\,\Gamma$) the bounded (resp. unbounded) component of $\mathbb{C} \setminus \Gamma$. Finally, for a symbol V, we denote $W(z) = V(z)e^{-|z|^2}$ and $V_\pm = (|V| \pm V)/2$.

2. Does T_V have infinitely many negative eigenvalues?

Here we address the following question. Let $V = V_+ - V_-$ be a bounded compactly supported function and suppose that neither V_+ nor V_- is identically equal to zero. What are sufficient conditions for T_V to have infinitely many negative eigenvalues? Of course, since $T_{-V} = -T_V$, this is equivalent to the same question about positive eigenvalues. The results presented here are by no means sharp; our purpose is merely to illustrate the fact that the finiteness of the number of negative eigenvalues of T_V depends upon the geometry of the sets $\text{supp}\,V_+$ and $\text{supp}\,V_-$.

2.1. **A "negative" example: the proof of Theorem 1.1.** Our reasoning is based on the formula

$$(T_V f, f)_{\mathcal{F}^2} = (T_{V_+} f, f)_{\mathcal{F}^2} \left(1 - \frac{(T_{V_-} f, f)_{\mathcal{F}^2}}{(T_{V_+} f, f)_{\mathcal{F}^2}}\right), \quad f \in \mathcal{F}^2 \tag{2.1}$$

and on an estimate for the quotient $(T_{V_-}f, f)_{\mathcal{F}^2}/(T_{V_+}f, f)_{\mathcal{F}^2}$. For any $f \in \mathcal{F}^2$ we have:

$$(T_{V_+}f, f)_{\mathcal{F}^2} = \int_{\mathrm{supp}\, V_+} W_+(z)|f(z)|^2 dm(z) \geq \int_{\mathrm{Ext}\,\Gamma} W_+(z)|f(z)|^2 dm(z). \qquad (2.2)$$

Next, let $\Gamma_\delta \subset \mathrm{Ext}\,\Gamma$ be a strip along Γ such that $W_+ \geq \delta/2$ in Γ_δ. Then, for some $C > 0$, we have

$$\int_{\mathrm{Ext}\,\Gamma} W_+(z)|f(z)|^2 dm(z) \geq \frac{\delta}{2} \int_{\Gamma_\delta} |f(z)|^2 dm(z) \geq C \int_\Gamma |f(z)|^2 d\ell(z) \qquad (2.3)$$

where $d\ell(z)$ is the length measure on the curve Γ. Next, the Cauchy integral formula can be written as

$$f(z) = \int_\Gamma K(z, \zeta)f(\zeta)d\ell(\zeta), \quad z \in \mathrm{supp}\, V_-,$$

where $K(z, \zeta)$ is smooth and bounded for $z \in \mathrm{supp}\, V_-$, $\zeta \in \Gamma$. Thus, the integral operator from $L^2(\Gamma, d\ell)$ to $L^2(\mathrm{supp}\, V_-, W_-(z)dm(z))$ with the kernel $K(z, \zeta)$ is compact. It follows that for any $\varepsilon > 0$ there exists a subspace $\mathcal{L}_\varepsilon \subset \mathcal{F}^2$ of a finite codimension such that for all $f \in \mathcal{L}_\varepsilon$ one has

$$(T_{V_-}f, f)_{\mathcal{F}^2} = \int_{\mathrm{supp}\, V_-} W_-(z)|f(z)|^2 dm(z) \leq \varepsilon \int_\Gamma |f(z)|^2 d\ell(z).$$

Recalling (2.1), (2.2), (2.3) and choosing $\varepsilon < \frac{2}{C}$, we get

$$(T_V f, f)_{\mathcal{F}^2} \geq \frac{1}{2}(T_{V_+}f, f)_{\mathcal{F}^2} > 0, \quad \forall f \in \mathcal{L}_\varepsilon.$$

Since the codimension of $\mathcal{L}_\varepsilon$ is finite, by the min-max principle it follows that there are at most finitely many negative eigenvalues of T_V. ∎

2.2. **Sufficient conditions: logarithmic capacity.** This section is based on sharp asymptotic eigenvalue estimates for Bargmann-Toeplitz operators with non-negative weight obtained in [4]. These estimates involve logarithmic capacity of the support of the symbol. The notion of the logarithmic capacity of a compact set in $\mathbb{C}$ is introduced in the framework of potential theory; for a detailed exposition, see e.g. [10]. Let us recall some basic properties of logarithmic capacity.

(i) If $K_1 \subset K_2 \subset \mathbb{C}$, then $\mathrm{Cap}\, K_1 \leq \mathrm{Cap}\, K_2$.

(ii) For a compact set $K \subset \mathbb{C}$, $\mathrm{Cap}\, K$ coincides with the logarithmic capacity of the outer boundary of K (i.e. the boundary of the unbounded component of K^c).

(iii) The logarithmic capacity of a disc of the radius R is R.

We denote

$$\mathrm{supp}_-(V) = \left\{ z \in \mathbb{C} \mid \limsup_{r \to +0} \frac{\log \int_{|\zeta - z| \leq r} |V(\zeta)|dm(\zeta)}{\log r} < \infty \right\}.$$

It is clear that $\mathrm{supp}_-(V) \subset \mathrm{supp}(V)$, and all the Lebesgue points z of V such that $V(z) > 0$ belong to $\mathrm{supp}_-(V)$.

Theorem 2.1. *Suppose that the symbol* $V = V_+ - V_-$ *with positive and negative parts* $V_+, V_- \geq 0$ *satisfies*

$$\mathrm{Cap}\,\mathrm{supp}_-(V_-) > \mathrm{Cap}\,\mathrm{supp}(V_+). \qquad (2.4)$$

Then the operator T_V has infinitely many negative eigenvalues.

We note that by the properties (i), (ii) of logarithmic capacity, under the hypothesis of Theorem 1.1 we necessarily have $\operatorname{Cap} \operatorname{supp}_-(V_-) \leq \operatorname{Cap} \operatorname{supp}(V_+)$. Thus, Theorem 1.1 and Theorem 2.1 are in agreement with each other.

In order to prove Theorem 2.1, we need to recall the results of [4] concerning the behavior of the spectrum of the Bargmann-Toeplitz operators T_V with non-negative compactly supported symbols V. Denote by $s_n = s_n(T_V)$ the singular numbers of the operator T_V for $V \geq 0$. Lemma 1 of [4] together with the estimates (1.15), (1.16) of [4] and the Stirling formula provides the estimates

$$e(\operatorname{Cap} \operatorname{supp}_-(V))^2 \leq \liminf_{n \to \infty} n s_n^{1/n} \leq \limsup_{n \to \infty} n s_n^{1/n} \leq e(\operatorname{Cap} \operatorname{supp}(V))^2. \tag{2.5}$$

Proof of Theorem 2.1. In order to simplify our notation, we will consider T_{-V} instead of T_V. That is, we assume that $\operatorname{Cap} \operatorname{supp}_-(V_+) > \operatorname{Cap} \operatorname{supp}(V_-)$ and prove that T_V has infinitely many positive eigenvalues. Take $\varepsilon > 0$ such that

$$a = e(\operatorname{Cap} \operatorname{supp}_-(V_+))^2 - \varepsilon > b = e(\operatorname{Cap} \operatorname{supp}(V_-))^2 + \varepsilon.$$

It follows from (2.5) that for m, n sufficiently large,

$$\lambda_n^+(T_{V_+}) \geq n^{-n} a^n, \quad \lambda_m^+(T_{V_-}) \leq m^{-m} b^m. \tag{2.6}$$

By the well known inequality for the eigenvalues of a sum of two compact operators (see e.g. [18, Section 95, formula (10)]) we have

$$\lambda_{n+m-1}^+(T_{V_+}) = \lambda_{n+m-1}^+(T_V + T_{V_-}) \leq \lambda_n^+(T_V) + \lambda_m(T_{V_-}),$$

which can be rewritten as

$$\lambda_n^+(T_V) \geq \lambda_{m+n-1}^+(T_{V_+}) - \lambda_m^+(T_{V_-}). \tag{2.7}$$

Below we choose an increasing sequence of integers $n_1, n_2, \ldots$ such that for each $n = n_m$ the r.h.s. of (2.7) is positive. This will prove that T_V has infinitely many positive eigenvalues.

By (2.6), it suffices to prove that for each m there exists n such that

$$(m + n - 1)^{-(m+n-1)} a^{m+n-1} > m^{-m} b^m. \tag{2.8}$$

An elementary analysis shows that if m is sufficiently large, then any $n < \gamma m / \log m$ with $\gamma < \log(a/b)$ satisfies (2.8). This proves the required statement. ∎

2.3. **Application of conformal mapping.** Theorem 1.1 shows that if $\operatorname{supp} V_-$ is "locked" inside $\operatorname{supp} V_+$, then T_V has finitely many negative eigenvalues. Here we show that in some sense the converse is true: if V is negative somewhere at the outer boundary of its support, then T_V has infinitely many negative eigenvalues.

We state the following result in its simplest form; it is easy to suggest various generalisations.

Theorem 2.2. *Suppose that V is continuous and the outer boundary of $\operatorname{supp} V$ is a smooth simple curve Γ. Suppose that for some $z_0 \in \Gamma$ and some $r > 0$, one has*

$$V(z) < 0 \text{ for all } z \in \operatorname{Int} \Gamma, \ |z - z_0| < r.$$

Then T_V has infinitely many negative eigenvalues.

Sketch of proof. The idea of the proof is to apply a conformal mapping which "blows up" the support of V_- and then to use Theorem 2.1. The proof also uses the reformulation of the problem in terms of Toeplitz operators in Bergman spaces. Without going into details, we describe the main steps of the proof.

Step 1: Let $\mathcal{D}$ be a simply connected domain in $\mathbb{C}$ with smooth boundary. The Bergman space $\mathcal{B}^2(\mathcal{D})$ is the subspace of $L^2(\mathcal{D}) = L^2(\mathcal{D}, dm)$ which consists of all functions analytic in $\mathcal{D}$. Let $P_{\mathcal{B}^2}$ be the orthogonal projection in $L^2(\mathcal{D})$ onto $\mathcal{B}^2(\mathcal{D})$. Then, for a bounded function F with $\operatorname{supp} F \subset \mathcal{D}$, the Bergman-Toeplitz operator $T_F(\mathcal{D})$ is the operator

$$T_F(\mathcal{D}) : \mathcal{B}^2(\mathcal{D}) \to \mathcal{B}^2(\mathcal{D}), \quad T_F(\mathcal{D})g = P_{\mathcal{B}^2}Fg, \quad g \in \mathcal{B}^2(\mathcal{D}).$$

The quadratic form of the operator $T_F(\mathcal{D})$ is given by

$$(T_F(\mathcal{D})g, g)_{\mathcal{B}^2(\mathcal{D})} = (Fg, g)_{L^2(\mathcal{D})}, \quad g \in \mathcal{B}^2(\mathcal{D}).$$

We use the following important observation (see [19] for the details). Since the set of all analytic polynomials is dense in $\mathcal{B}^2(\mathcal{D})$, by the min-max principle it follows that the negative spectrum of $T_F(\mathcal{D})$ is infinite if and only if for any $n \in \mathbb{N}$ there exists a linear set $\mathcal{L}_n$, $\dim \mathcal{L}_n = n$ of analytic polynomials such that $(Fg, g)_{L^2(\mathcal{D})} < 0$ for all $g \in \mathcal{L}_n$, $g \neq 0$. Note that for any polynomial g the value of the quadratic form $(Fg, g)_{L^2(\mathcal{D})}$ depends on F but not on the choice of the domain $\mathcal{D}$ as long as $\operatorname{supp} F \subset \mathcal{D}$. Thus, the above stated necessary and sufficient condition of the infiniteness of the negative spectrum of $T_F(\mathcal{D})$ is in fact independent of $\mathcal{D}$.

This reasoning can also be applied to the Bargmann-Toeplitz operator T_V. We arrive at the following conclusion (recall that $W(z) = V(z)e^{-|z|^2}$): *the Bargmann-Toeplitz operator T_V in $\mathcal{F}^2$ has infinitely many negative eigenvalues if and only if for some domain $\mathcal{D} \supset \operatorname{supp} V$ the Bergman-Toeplitz operator $T_W(\mathcal{D})$ in $\mathcal{B}^2(\mathcal{D})$ has infinitely many negative eigenvalues.*

Step 2: Under the hypothesis of Theorem 2.2, let us fix a point z_1 outside the support of V and choose a simply connected domain $\mathcal{D}$ with smooth boundary such that $\operatorname{supp} V \subset \mathcal{D}$ and $z_1 \notin \overline{\mathcal{D}}$ (here $\overline{\mathcal{D}}$ is the closure of $\mathcal{D}$). Consider the conformal map $w = \varphi(z) = (z - z_1)^{-1}$ and let $\mathcal{D}' = \varphi(\mathcal{D})$. The map φ generates a unitary map Φ from $L^2(\mathcal{D})$ to $L^2(\mathcal{D}')$:

$$\Phi : f(z) \mapsto (\Phi f)(w) = f(\varphi^{-1}(w))|\varphi'(\varphi^{-1}(w))|^{-1/2}.$$

We have $\Phi T_W(\mathcal{D})\Phi^* = T_{W'}(\mathcal{D}')$ with $W' = W \circ \varphi$. Thus, the operator $T_W(\mathcal{D})$ in $\mathcal{B}^2(\mathcal{D})$ has infinitely many negative eigenvalues if and only if the operator $T_{W'}(\mathcal{D}')$ in $\mathcal{B}^2(\mathcal{D}')$ has infinitely many negative eigenvalues. Combining this with the previous step, we obtain that *the operator T_V in $\mathcal{F}^2$ has infinitely many negative eigenvalues if and only if the operator $T_{V'}$ in $\mathcal{F}^2$ has infinitely many negative eigenvalues, where $V'(z) = e^{|z|^2}W'(z)$.*

Step 3: Now it remains to choose z_1 in such a way that

$$\operatorname{Cap} \operatorname{supp}_-(V'_-) > \operatorname{Cap} \operatorname{supp}(V'_+); \tag{2.9}$$

then Theorem 2.1 can be applied to $T_{V'}$. Clearly, we have $\operatorname{supp}(V'_+) = \varphi(\operatorname{supp} V_+)$. Next, let $\Omega_- = \{z \in \mathbb{C} \mid V(z) < 0\}$; by the continuity of V, we have

$$\Omega_- \subset \operatorname{supp}_-(V_-) \quad \text{and} \quad \varphi(\Omega_-) \subset \operatorname{supp}_-(V'_-).$$

Now it is easy to see that by choosing z_1 sufficiently close to z_0, we can ensure that

$$\operatorname{Cap} \varphi(\Omega_-) > \operatorname{Cap} \varphi(\operatorname{supp} V_+),$$

and thus (2.9) holds true. It follows that $T_{V'}$, and therefore T_V has infinitely many negative eigenvalues. ∎

3. Symbols with separated positive and negative parts

In this section we prove Theorems 1.2 and 1.3.

3.1. **"Outbedding" operators and $A_n^{\pm}$.** Throughout this section, we assume that Ω_+ and Ω_- are Lipschitz domains which satisfy (1.7). We start with some notation. For $n \in \mathbb{N}$ we denote by $\mathcal{P}_n$ the space of all polynomials in z of degree $\leq n$. The $(n+1)$-dimensional linear space $\mathcal{P}_n$ can be endowed with the inner product structure of $L^2(\Omega_{\pm})$. We denote the resulting (finite dimensional) Hilbert spaces by $\mathcal{P}_n^{\pm}$. Consider the operators

$$S_n^+ : \mathcal{P}_n^+ \to \mathcal{P}_n^-, \quad S_n^+ f = f \quad \text{and} \quad S_n^- : \mathcal{P}_n^- \to \mathcal{P}_n^+, \quad S_n^- f = f.$$

Thus, S_n^+, S_n^- map each polynomial to itself. By analogy with embedding operators, we will call S_n^+, S_n^- *the outbedding operators.* We define the operators $A_n^{\pm}$ in $\mathcal{P}_n^{\pm}$ as

$$A_n^{\pm} = (S_n^{\pm})^* S_n^{\pm}.$$

The quadratic forms of these operators are

$$(A_n^+ f, f)_+ = \|f\|_-^2, \quad f \in \mathcal{P}_n^+ \quad \text{and} \quad (A_n^- f, f)_- = \|f\|_+^2, \quad f \in \mathcal{P}_n^-.$$

By definition, we have

$$\frac{(A_n^+ f, f)_+}{\|f\|_+^2} = \frac{\|f\|_-^2}{(A_n^- f, f)_-}, \quad f \in \mathcal{P}_n,$$

and therefore, by the min-max principle,

$$\dim \operatorname{Ker}(A_n^+ - \lambda I) = \dim \operatorname{Ker}(A_n^- - \lambda^{-1} I), \quad \lambda > 0.$$

It follows that

$$N((\lambda, \infty); A_n^{\pm}) = N((0, \lambda^{-1}); A_n^{\mp}), \quad \lambda > 0. \tag{3.1}$$

In our discussion of the spectrum of Toeplitz operators, the distribution of singular values of the outbedding operators plays a crucial role.

Lemma 3.1. *For any $0 < a_1 < a_2 < \infty$,*

$$N((a_1, a_2); A_n^{\pm}) = O(1) \text{ as } n \to \infty. \tag{3.2}$$

Proof. 1. Let us choose disjoint closed sets Ω_+' and Ω_-' whose boundaries are simple Jordan curves and such that $\Omega_{\pm} \subset \Omega_{\pm}'$. Consider the function $\phi(z)$ defined on $\Omega_+' \cup \Omega_-'$, such that $\phi(z) = 1$ for $z \in \Omega_+'$ and $\phi(z) = 0$ for $z \in \Omega_-'$. By Theorem 6, Chapter 2 in [24], the function ϕ can be approximated by polynomials uniformly on $\Omega_+' \cup \Omega_-'$. In particular, for any $\gamma > 0$ there exists a polynomial $p \not\equiv 0$ such that

$$\sup_{\Omega_-} |p(z)| \leq \gamma \inf_{\Omega_+} |p(z)|. \tag{3.3}$$

Let us fix a polynomial p which satisfies (3.3) with $\gamma^2 = a_1/a_2$ and set $m = \deg p$.

2. Consider the subspace $\mathcal{L}_n = \operatorname{Ran} E([0, a_2], A_n^+) \subset \mathcal{P}_n^+$. Then

$$\|f\|_-^2 \leq a_2 \|f\|_+^2, \quad \forall f \in \mathcal{L}_n. \tag{3.4}$$

Denote $\mathcal{M}_{n+m} = \{pf : f \in \mathcal{L}_n\} \subset \mathcal{P}_{n+m}^+$. Then for any $g \in \mathcal{M}_{m+n}$, we have

$$\frac{\|g\|_-^2}{\|g\|_+^2} = \frac{\|pf\|_-^2}{\|pf\|_+^2} \leq \frac{\|f\|_-^2 \sup_{\Omega_-} |p|^2}{\|f\|_+^2 \inf_{\Omega_+} |p|^2} \leq \gamma^2 a_2 = a_1.$$

It follows that

$$N([0, a_1]; A_{m+n}^+) \geq \dim \mathcal{M}_{n+m} = \dim \mathcal{L}_n = N([0, a_2]; A_n^+). \tag{3.5}$$

3. Since $\dim \mathcal{P}_n^+ = n + 1$, the inequality (3.5) can be written as

$$(n + m + 1) - N((a_1, \infty); A_{n+m}^+) \geq (n + 1) - N((a_2, \infty); A_n^+). \tag{3.6}$$

By variational considerations, we have

$$N((a_1, \infty); A_n^+) \leq N((a_1, \infty); A_{n+m}^+).$$

Combining this with (3.6), we get

$$N((a_1, \infty); A_n^+) - N((a_2, \infty); A_n^+) \leq m,$$

which implies $N((a_1, a_2); A_n^+) \leq m$. ∎

In the next lemma we estimate the norm of the outbedding operators $S_n^\pm$ or, equivalently, the norm of the operators $A_n^\pm$. In order to do this, and for our further analysis, we need to recall some facts from [23] related to the theory of orthogonal polynomials.

Let $P_n^\pm$ be the n-th degree orthogonal polynomial corresponding to the measure $\chi_{\Omega_\pm}(z)dm(z)$ in $\mathbb{C}$. We assume that $P_n^\pm$ is normalised such that $\|P_n^\pm\|_\pm = 1$. Let $g_\pm$ be the Green's function corresponding to the domain $\Omega_\pm^c$; this function is uniquely defined by the requirements that $g_\pm$ is harmonic in $\Omega_\pm^c$, vanishes on $\Omega_\pm$ and $g_\pm(z) - \log|z| = O(1)$ as $|z| \to \infty$. We will use the following estimates from [23, Theorem 1.1.4 and Lemma 1.1.7]:

$$\limsup_{n \to \infty} |P_n^\pm(z)|^{1/n} \leq e^{g_\pm(z)} \tag{3.7}$$

locally uniformly in $\mathbb{C}$,

$$\liminf_{n \to \infty} |P_n^\pm(z)|^{1/n} \geq e^{g_\pm(z)} \tag{3.8}$$

locally uniformly in $\mathbb{C} \setminus \mathrm{Co}(\Omega_\pm)$. We will denote

$$a_+ = \sup_{\Omega_-} g_+, \qquad a_- = \inf_{\Omega_+} g_-. \tag{3.9}$$

Lemma 3.2. *Let Ω_+ and Ω_- be domains with Lipschitz boundaries such that $\mathrm{Co}(\Omega_+) \cap \mathrm{Co}(\Omega_-) = \varnothing$. Then, for any $b_+ > a_+$ one has*

$$\|S_n^+\| = \|A_n^+\|^{1/2} \leq e^{b_+ n} \tag{3.10}$$

for all sufficiently large n.

Proof. It follows from (3.7) that $\|P_n^+\|_- \leq Ce^{b_+ n}$ for some $C > 0$ and all n. Let q be a polynomial in $\mathcal{P}_n$. We expand q in the basis of orthogonal polynomials P_k^+, $k = 0, \ldots, n$:

$q = \sum_{k=0}^{n} c_k P_k^+$. Then we have

$$\|q\|_-^2 \leq \left(\sum_{k=0}^{n} |c_k| \|P_k^+\|_-\right)^2 \leq \max_{k=0,\ldots,n} \|P_k^+\|_-^2 \left(\sum_{k=0}^{n} |c_k|\right)^2 \leq C^2 e^{2b_+ n} \left(\sum_{k=0}^{n} |c_k|\right)^2$$

$$\leq C^2 e^{2b_+ n}(n+1) \sum_{k=0}^{n} |c_k|^2 = C^2 e^{2b_+ n}(n+1) \|q\|_+^2.$$

Thus, for all sufficiently large n we have

$$\|q\|_- \leq C e^{b_+ n} \sqrt{n+1} \|q\|_+ \leq C_\varepsilon e^{(b_+ + \varepsilon)n} \|q\|_+$$

where $\varepsilon > 0$ can be chosen arbitrary small. This proves the required statement. ∎

Lemma 3.3. *Denote*

$$\delta_\pm = \liminf_{n\to\infty} \frac{1}{n} N((0,1); A_n^\pm), \quad \Delta_\pm = \limsup_{n\to\infty} \frac{1}{n} N((0,1); A_n^\pm).$$

Then we have

$$0 < \delta_\pm \leq \Delta_\pm < 1 \tag{3.11}$$

and

$$\delta_- + \Delta_+ = 1, \quad \delta_+ + \Delta_- = 1. \tag{3.12}$$

Proof. 1. Let a_+, a_- be as in (3.9) and let $b_\pm$ be positive constants such that $b_+ > a_+$ and $b_- < a_-$. Let q be a polynomial with $\deg q \leq k$. Using (3.7) and (3.10), we obtain that for any $\delta > 0$ and for all sufficiently large k and m,

$$\|P_m^- q\|_- \leq e^{\delta m} \|q\|_- \leq e^{\delta m + b_+ k} \|q\|_+. \tag{3.13}$$

On the other hand, by (3.8), for all sufficiently large m,

$$\|P_m^- q\|_+ \geq e^{b_- m} \|q\|_+. \tag{3.14}$$

Combining (3.13) and (3.14), we obtain that for any $b_+ > a_+$ and $b_- < a_-$ and all sufficiently large m and k we have

$$\|P_m^- q\|_- \leq e^{b_+ k - b_- m} \|P_m^- q\|_+; \tag{3.15}$$

here δ has been incorporated into b_-.

2. Let us choose $\alpha \in (\frac{a_+}{a_+ + a_-}, 1)$ and set $\beta = 1 - \alpha$. Consider the subspace

$$\mathcal{L}_n = \{P_{[\alpha n]}^- q : \deg q < [\beta n]\} \subset \mathcal{P}_n, \quad \dim \mathcal{L}_n = [\beta n].$$

By (3.15), we have for any $b_- < a_-$ and any $b_+ > a_+$ and all sufficiently large n,

$$\|f\|_- \leq e^{b_+ [\beta n] - b_- [\alpha n]} \|f\|_+, \quad \forall f \in \mathcal{L}_n.$$

We have $b_+ [\beta n] - b_- [\alpha n] \leq (b_+ \beta - b_- \alpha)n + b_-$. By our choice of α and β, we have $a_+ \beta - a_- \alpha < 0$, and therefore the expression $(b_+ \beta - b_- \alpha)$ can be made negative by a suitable choice of $b_\pm$. Thus we obtain that for all sufficiently large n,

$$\|f\|_- < \|f\|_+, \quad \forall f \in \mathcal{L}_n, \quad f \neq 0. \tag{3.16}$$

3. From (3.16) it follows that $N((0,1); A_n^+) \geq \dim \mathcal{L}_n = [\beta n]$, and so $\delta_+ \geq \beta > 0$. Interchanging Ω_+ and Ω_-, we also get $\delta_- > 0$. Using (3.1) and Lemma 3.1, we obtain

$$0 < \delta_- = \liminf_{n\to\infty} n^{-1} N((0,1); A_n^-) = \liminf_{n\to\infty} n^{-1} N((1,\infty); A_n^+)$$

$$= \liminf_{n\to\infty} n^{-1}\big(n + 1 - N((0,1); A_n^+) - N(\{1\}; A_n^+)\big) = 1 - \Delta_+,$$

and so $\Delta_+ < 1$. In a similar way, one proves that $\delta_+ = 1 - \Delta_-$ and so $\Delta_- < 1$. ∎

3.2. **Estimates for Toeplitz operators.** In order to prove Theorem 1.2, we first establish some relations between the spectral distribution functions of T_V and $A_n^\pm$. We need two elementary auxiliary statements. We denote $D_r(a) = \{z \in \mathbb{C} \mid |z - a| < r\}$ and

$$z^n \mathcal{F}^2 = \{f \in \mathcal{F}^2 \mid f(z)z^{-n} \in \mathcal{F}^2\}, \quad n \in \mathbb{N}.$$

Lemma 3.4. *(i) For any $r > 0$ there exists a constant $\alpha > 0$ such that for all sufficiently large n, one has*

$$\|f\|_{L^2(D_r(0))}^2 \leq e^{-n\log n + \alpha n}\|f\|_{\mathcal{F}^2}^2, \quad \forall f \in z^n \mathcal{F}^2.$$

(ii) For any $a \in \mathbb{C}$ and any $r > 0$ there exists a constant $\alpha > 0$ such that for all sufficiently large n,

$$\|f\|_{L^2(D_r(a))}^2 \geq e^{-n\log n - \alpha n}\|f\|_{\mathcal{F}^2}^2, \quad \forall f \in \mathcal{P}_n.$$

Proof. (i) Write $f \in z^n \mathcal{F}^2$ as $f(z) = \sum_{j=n}^\infty f_j z^j$. We get

$$\|f\|_{L^2(D_r(0))}^2 = \pi \sum_{j=n}^\infty |f_j|^2 \frac{r^{2j+2}}{j+1},$$

$$\|f\|_{\mathcal{F}^2}^2 = \sum_{j=n}^\infty |f_j|^2 \|z^j\|_{\mathcal{F}^2}^2 = \pi \sum_{j=n}^\infty |f_j|^2 j!$$

and therefore, for all sufficiently large n,

$$\|f\|_{L^2(D_r(0))}^2 = \pi \sum_{j=n}^\infty |f_j|^2 j! \frac{r^{2j+2}}{(j+1)!} \leq \frac{r^{2n+2}}{(n+1)!}\|f\|_{\mathcal{F}^2}^2.$$

Combining this with the Stirling formula, we obtain the required estimate.

(ii) Without the loss of generality assume $r \leq 1$. Write $f \in \mathcal{P}_n$ as $f(z) = \sum_{j=0}^n f_j(z-a)^j$. Then

$$\|f\|_{L^2(D_r(a))}^2 = \pi \sum_{j=0}^n |f_j|^2 \frac{r^{2j+2}}{j+1},$$

$$\|f\|_{\mathcal{F}^2} \leq \sum_{j=0}^n |f_j|\|(z-a)^j\|_{\mathcal{F}^2} \leq \max_{k=0,\ldots,n}\|(z-a)^k\|_{\mathcal{F}^2} \sum_{j=0}^n |f_j|.$$

An elementary calculation shows that $\|(z-a)^k\|^2_{\mathcal{F}^2} \leq e^{\gamma k} k!$ for some $\gamma > 0$ and all $k \in \mathbb{N}$. Thus,

$$\|f\|^2_{\mathcal{F}^2} \leq e^{\gamma n} n! \left(\sum_{j=0}^n |f_j|\right)^2 \leq e^{\gamma n} n! \left(\sum_{j=0}^n |f_j| \frac{r^{j+1}}{\sqrt{j+1}}\right)^2 (n+1) r^{-2n-2}$$

$$\leq e^{\gamma n} n! \left(\sum_{j=0}^n |f_j|^2 \frac{r^{2j+2}}{j+1}\right) (n+1)^2 r^{-2n-2} \leq e^{\gamma n} n! \pi^{-1} \|f\|^2_{L^2(D_r(a))} (n+1)^2 r^{-2n-2}.$$

Combining this with the Stirling formula, we get the required statement. ∎

Recall that by our assumptions, the symbol V is given by the formula (1.8), where the functions $v_\pm$ are bounded, positive, and separated from zero. Let us write

$$W(z) = W_+(z) - W_-(z) = w_+(z)\chi_{\Omega_+}(z) - w_-(z)\chi_{\Omega_-}(z),$$

where the functions $w_\pm$ satisfy

$$0 < \tau_\pm \leq w_\pm(z) \leq \sigma_\pm \tag{3.17}$$

with some constants $\tau_\pm$ and $\sigma_\pm$.

Lemma 3.5. *For all sufficiently large α, all sufficiently small $\varkappa \in (0,1)$ and all sufficiently large n, one has*

$$N((e^{-n \log n} e^{-\alpha n}, \infty); T_V) \geq N((0, \varkappa); A_n^+). \tag{3.18}$$

Proof. Choose $\varkappa$ sufficiently small so that $\gamma = \frac{\sigma_-}{\tau_+}\varkappa < 1$. For all f in the subspace $\mathrm{Ran}\, E((0, \varkappa); A_n^+) \subset \mathcal{P}_n^+$ we have

$$(W_- f, f)_- \leq \sigma_- \|f\|^2_- < \sigma_- \varkappa \|f\|^2_+ \leq \frac{\sigma_- \varkappa}{\tau_+}(W_+ f, f)_+ = \gamma(W_+ f, f)_+.$$

Therefore

$$\frac{(T_V f, f)_{\mathcal{F}^2}}{\|f\|^2_{\mathcal{F}^2}} = \frac{(W_+ f, f)_+ - (W_- f, f)_-}{\|f\|^2_{\mathcal{F}^2}} \geq (1-\gamma)\frac{(W_+ f, f)_+}{\|f\|^2_{\mathcal{F}^2}} \geq (1-\gamma)\tau_+ \frac{\|f\|^2_+}{\|f\|^2_{\mathcal{F}^2}} \tag{3.19}$$

for all $f \in \mathrm{Ran}\, E((0, \varkappa); A_n^+)$. Choose a disk $D_r(a) \subset \Omega_+$; then $\|f\|_+ \geq \|f\|_{L^2(D_r(a))}$. Now combining (3.19) with Lemma 3.4(ii), we obtain

$$(T_V f, f)_{\mathcal{F}^2} \geq e^{-n \log n} e^{-\alpha n} \|f\|^2_{\mathcal{F}^2}, \quad \forall f \in \mathrm{Ran}\, E((0, \varkappa); A_n^+),$$

for some constant $\alpha > 0$ and all sufficiently large n. Applying the min-max principle, we obtain the required result. ∎

Lemma 3.6. *For all sufficiently large α, all sufficiently small $\varkappa$ and all sufficiently large n one has*

$$N((e^{-n \log n} e^{\alpha n}, \infty); T_V) \leq N((0, \varkappa^{-1}); A_n^+). \tag{3.20}$$

Proof. 1. Note that the subspace $z^{n+1}\mathcal{F}^2$ is orthogonal to $\mathcal{P}_n$ in $\mathcal{F}^2$ and therefore these subspaces are linearly independent. Consider the direct sum of subspaces

$$\mathcal{L}_n = \left(\mathrm{Ran}\, E([\varkappa^{-1}, \infty), A_n^+)\right) \dot{+} (z^{n+1}\mathcal{F}^2)$$

The codimension of $\mathcal{L}_n$ in $\mathcal{F}^2$ equals $(n+1) - N([\varkappa^{-1}, \infty); A_n^+) = N((0, \varkappa^{-1}); A_n^+)$.

Let $\phi \in \mathcal{L}_n$, $\phi = f + g$, $f \in \operatorname{Ran} E([\varkappa^{-1}; \infty), A_n^+)$, $g \in z^{n+1}\mathcal{F}^2$, so $\|f\|_+^2 \leq \varkappa \|f\|_-^2$ and $\|\phi\|_{\mathcal{F}^2}^2 = \|f + g\|_{\mathcal{F}^2}^2 = \|f\|_{\mathcal{F}^2}^2 + \|g\|_{\mathcal{F}^2}^2 \geq \|g\|_{\mathcal{F}^2}^2$. Using the Cauchy-Schwartz inequality and (3.17), we get

$$(T_{V_+}\phi, \phi)_{\mathcal{F}^2} = (W_+\phi, \phi)_+ \leq \sigma_+ \|f + g\|_+^2 \leq 2\sigma_+ \|f\|_+^2 + 2\sigma_+ \|g\|_+^2;$$

$$(T_{V_-}\phi, \phi)_{\mathcal{F}^2} = (W_-\phi, \phi)_- \geq \tau_- \|f + g\|_-^2$$
$$\geq \tau_-(\|f\|_-^2 + \|g\|_-^2 - \epsilon \|f\|_-^2 - \epsilon^{-1}\|g\|_-^2).$$

Combining the above estimates and using $\|f\|_+^2 \leq \varkappa \|f\|_-^2$, we obtain

$$(T_V\phi, \phi)_{\mathcal{F}^2} \leq 2\sigma_+ \|g\|_+^2 + \tau_-(\epsilon^{-1} - 1)\|g\|_-^2 - (\tau_- - 2\varkappa\sigma_+ - \tau_-\epsilon)\|f\|_-^2.$$

Now we suppose that $\varkappa$ and ϵ are chosen sufficiently small so that $\tau_- - 2\varkappa\sigma_+ - \tau_-\epsilon \geq 0$. Then

$$(T_V\phi, \phi)_{\mathcal{F}^2} \leq 2\sigma_+ \|g\|_+^2 + \tau_-(\epsilon^{-1} - 1)\|g\|_-^2 \leq 2\sigma_+ \|g\|_+^2 + \tau_-\epsilon^{-1}\|g\|_-^2.$$

In this way we have established the inequality

$$\frac{(T_V\phi, \phi)_{\mathcal{F}^2}}{\|\phi\|_{\mathcal{F}^2}^2} \leq C\frac{\|g\|_+^2 + \|g\|_-^2}{\|g\|_{\mathcal{F}^2}^2}, \quad \forall \phi = f + g \in \mathcal{L}_n \tag{3.21}$$

with $C = \max\{2\sigma_+, \tau_-\epsilon^{-1}\}$.

2. Suppose that r is chosen so large that $\Omega_+ \cup \Omega_- \subset D_r(0)$. Then $\|g\|_+^2 + \|g\|_-^2 \leq \|g\|_{L^2(D_r(0))}^2$, and, therefore, by Lemma 3.4(i) and (3.21),

$$\frac{(T_V\phi, \phi)_{\mathcal{F}^2}}{\|\phi\|_{\mathcal{F}^2}^2} \leq e^{-n\log n}e^{\alpha n}, \quad \forall \phi \in \mathcal{L}_n$$

for all sufficiently large n. It follows now by the min-max principle that

$$N((e^{-n\log n}e^{\alpha n}, \infty); T_V) \leq \operatorname{codim} \mathcal{L}_n = N((0, \varkappa^{-1}); A_n^+),$$

for all sufficiently large n, as required. ∎

3.3. Proof of Theorems 1.2 and 1.3.

Proof of Theorem 1.2. 1. Let us prove the asymptotics (1.5). From Lemmas 3.5, 3.6, by interchanging Ω_+ and Ω_-, and using (3.1) we obtain

$$N((e^{-n\log n}e^{-\alpha n}, \infty); -T_V) \geq N((0, \varkappa); A_n^-) = N((\varkappa^{-1}, \infty); A_n^+) \tag{3.22}$$

$$N((e^{-n\log n}e^{\alpha n}, \infty); -T_V) \leq N((0, \varkappa^{-1}); A_n^-) = N((\varkappa, \infty); A_n^+). \tag{3.23}$$

Combining (3.18) and (3.22), we obtain

$$N((e^{-n\log n}e^{-\alpha n}, \infty); |T_V|) = N((e^{-n\log n}e^{-\alpha n}, \infty); T_V) + N((e^{-n\log n}e^{-\alpha n}, \infty); -T_V)$$
$$\geq N((0, \varkappa); A_n^+) + N((\varkappa^{-1}, \infty); A_n^+) = (n + 1) - N([\varkappa, \varkappa^{-1}]; A_n^+). \tag{3.24}$$

In the same way, combining (3.20) with (3.23), we get

$$N((e^{-n\log n}e^{\alpha n}, \infty); |T_V|)$$
$$\leq N((\varkappa, \infty); A_n^+) + N((0, \varkappa^{-1}); A_n^+) = (n + 1) + N((\varkappa, \varkappa^{-1}), A_n^+). \tag{3.25}$$

Estimates (3.24), (3.25), together with Lemma 3.1, yield

$$s_{n-m}(T_V) \geq e^{-n\log n}e^{-\alpha n},$$
$$s_{n+m}(T_V) \leq e^{-n\log n}e^{\alpha n},$$

for all sufficiently large n and some fixed m. This proves (1.5).

2. Let us prove the '+' version of the asymptotic estimates (1.9). In order to do this, we combine Lemmas 3.1 and 3.3 with Lemma 3.5. We obtain that

$$N((e^{-n\log n}e^{-\alpha n}, \infty); T_V) \geq \delta_+ n + o(n), \quad n \to \infty.$$

The latter inequality can be re-written as

$$\lambda^+_{\delta_+ n + o(n)}(T_V) \geq e^{-n\log n}e^{-\alpha n}, \quad n \to \infty \tag{3.26}$$

which is equivalent to the first inequality in (1.9). Similarly, from Lemmas 3.1, 3.3 and 3.6, we obtain

$$\lambda^+_{\Delta_+ n + o(n)}(T_V) \leq e^{-n\log n}e^{\alpha n}, \quad n \to \infty \tag{3.27}$$

which gives the second inequality in (1.9). The '−' version of the asymptotic estimates (1.9) is obtained by applying the same reasoning to T_{-V}. ∎

By simple variational considerations we can now show that the estimates (1.9) hold under considerably weaker conditions.

Corollary 3.7. *Let the symbol V be a real bounded function with compact support. Suppose that for some $\varepsilon > 0$ there exists a closed set K of positive measure such that $V(z) \geq \varepsilon$ on K and $\mathrm{Co}(K) \cap \mathrm{Co}(\mathrm{supp}\, V_-) = \varnothing$. Then for the positive eigenvalues of T_V the estimate*

$$\lambda^+_n(T_V) \geq e^{-\beta n \log n}$$

holds true for some $\beta < \infty$ for all sufficiently large n.

Proof. Let us compare T_V with the Toeplitz operator T_F, where $F(z) = \varepsilon$ on K, $F(z) = \inf V$ on $\mathrm{supp}\, V_-$, and $F(z) = 0$ elsewhere. Since $F \leq V$, we have $T_F \leq T_V$. Applying Theorem 1.2 to T_F, we get the required result for T_V by variational considerations. ∎

Proof of Theorem 1.3. By symmetry, A^+_n and A^-_n are unitarily equivalent. Using this and (3.1), we obtain

$$N((0,1); A^+_n) = N((0,1); A^-_n) = N((1,\infty); A^+_n),$$

and therefore, by Lemma 3.1,

$$2N((0,1); A^\pm_n) = n - N(\{1\}; A^+_n) = n + O(1), \quad n \to \infty,$$

and so $N((0,1); A^\pm_n) = \frac{n}{2} + O(1)$. From here, following the same reasoning as in the second part of the proof of Theorem 1.2, we obtain the required estimates (1.12). ∎

4. Applications to the perturbed Landau Hamiltonian

4.1. **The Landau Hamiltonian.** The unperturbed Landau Hamiltonian H_0 is the Schrödinger operator with a constant homogeneous magnetic field B:

$$H_0 = \left(-i\tfrac{\partial}{\partial x_1} + \tfrac{B}{2}x_2\right)^2 + \left(-i\tfrac{\partial}{\partial x_2} - \tfrac{B}{2}x_1\right)^2 - B \quad \text{in } L^2(\mathbb{R}^2, dx_1 dx_2). \tag{4.1}$$

Here the constant term $-B$ is included for normalization reasons. In what follows we take $B = 2$; the general case can always be reduced to this one by scaling. It is well known that the spectrum of H_0 consists of the eigenvalues 0, 2, 4,... of infinite multiplicity; these eigenvalues are called the Landau levels.

Next, let $V : \mathbb{R}^2 \to \mathbb{R}$, $V(x) \to 0$ as $|x| \to \infty$. Consider the operator $H = H_0 + V$; here V stands for the operator of multiplication by the function $V = V(x)$ and has the meaning of the perturbing electric potential. As it is well known, the operator V is H_0-form compact and so the essential spectra of H_0 and H coincide. Thus, the spectrum of H consists of eigenvalues, and the only possible points of accumulation of these eigenvalues are the Landau levels. One can say that the perturbation 'splits' the Landau levels into 'clusters' of eigenvalues. Here we discuss the rate of accumulation of the eigenvalues in these clusters towards the Landau levels. For simplicity, we discuss only the case of the lowest Landau level 0. Without going into details we mention that combining the technique of [4] with the technique of this paper, it is not difficult to obtain similar results for the splitting of the higher Landau levels.

4.2. **Reduction to $P_0 V^\pm P_0$.** For $n \geq 0$, let P_n be the spectral projection of H_0 corresponding to the Landau level $2n$, i.e. $P_n = E(\{2n\}; H_0)$. It is common wisdom that the asymptotic behavior of the eigenvalues of H in the cluster around the Landau level $2n$ is determined by the asymptotics of the eigenvalues of the operator $P_n V P_n$. Results of this kind have been widely used in the case of a sign-definite V, see e.g. [15, 9, 17, 13, 4, 20]. Our situation, with V changing sign, is somewhat more complicated. The Theorem to follow expresses the above relation in exact terms. As mentioned above, we only discuss the case $n = 0$. We fix $\epsilon > 0$ and denote $V_\epsilon^\pm = V \pm \epsilon |V|$.

Theorem 4.1. *(i) There exists a constant m (which may depend on V and on ϵ) such that for any $\lambda < 0$,*

$$N((-\infty, \lambda); P_0 V P_0) \leq N((-\infty, \lambda); H) \leq N((-\infty, \lambda); P_0 V_\epsilon^- P_0) + m. \tag{4.2}$$

(ii) For any $a \in (0, 2)$, there exists a constant m (which may depend on V, ϵ and on a), such that for any $\lambda \in (0, a)$

$$N((\lambda, \infty); P_0 V_\epsilon^- P_0) - m \leq N((\lambda, a), H) \leq N((\lambda, \infty); P_0 V_\epsilon^+ P_0) + m. \tag{4.3}$$

Similar results were obtained by different methods in [15, 17, 13].

The key step in the proof of Theorem 4.1 is the following lemma. The idea of this lemma is borrowed from [9].

Lemma 4.2. *The following operator inequalities hold in the quadratic form sense:*

$$H_\epsilon^- \leq H \leq H_\epsilon^+,$$

where

$$H_\epsilon^\pm = H_0 + P_0(V \pm \epsilon |V|)P_0 + P_0^\perp(V \pm \epsilon^{-1}|V|)P_0^\perp,$$

and $P_0^\perp = I - P_0$.

Proof. This is a direct calculation. Let $\psi \in L^2(\mathbb{R}^2), \psi_0 = P_0\psi_0, \; \psi_1 = P_0^\perp\psi_0$. Then

$$(V\psi, \psi) \leq (V\psi_0, \psi_0) + (V\psi_1, \psi_1) + 2|(V\psi_0, \psi_1)|;$$

$$2|(V\psi_0, \psi_1)| \leq 2\||V|^{1/2}\psi_0\|\||V|^{1/2}\psi_1\| \leq \epsilon\||V|^{\frac{1}{2}}\psi_0\|^2 + \epsilon^{-1}\||V|^{\frac{1}{2}}\psi_1\|^2$$
$$= \epsilon(|V|\psi_0, \psi_0) + \epsilon^{-1}(|V|\psi_1, \psi_1),$$

and this proves the upper bound. The lower bound is proved in the same way. ∎

Proof of Theorem 4.1(i). The lower bound in (4.2) follows trivially from the min-max principle:

$$N((-\infty, \lambda); H) \geq N((-\infty, \lambda); P_0 H P_0) = N((-\infty, \lambda); P_0 V P_0),$$

since $P_0 H_0 P_0 = 0$.

To prove the upper estimate, we note that by the min-max principle,

$$H_\epsilon^- \leq H \quad \Rightarrow \quad N((-\infty, \lambda); H) \leq N((-\infty, \lambda); H_\epsilon^-),$$

and

$$N((-\infty, \lambda); H_\epsilon^-) = N((-\infty, \lambda); P_0 V_\epsilon^- P_0) + N((-\infty, \lambda); P_0^\perp(H_0 + V - \epsilon^{-1}|V|)P_0^\perp)$$
$$\leq N((-\infty, \lambda); P_0 V_\epsilon^- P_0) + m,$$

where $m = N((-\infty, 0); P_0^\perp(H_0 + V - \epsilon^{-1}|V|)P_0^\perp)$. Note that the quantity m is finite. Indeed, the bottom of the essential spectrum of the operator $P_0^\perp H_0 P_0^\perp|_{\mathrm{Ran}\, P_0^\perp}$ is 2, and $P_0^\perp(V - \epsilon^{-1}|V|)P_0^\perp$ is a relatively compact perturbation of this operator. Thus, the operator $P_0^\perp(H_0 + V - \epsilon^{-1}|V|)P_0^\perp$ has finitely many negative eigenvalues. ∎

In order to prove the second part of Theorem 4.1, we will use some machinery from [14]. The argument below has a general operator theoretic nature and applies to any pair of self-adjoint operators H_0, H which are semi-bounded from below and such that the difference $H - H_0$ is H_0-form compact. Under this assumption, the essential spectra of H_0 and H coincide. Consider the difference between the eigenvalue counting functions of H and H_0:

$$N((-\infty, \lambda); H_0) - N((-\infty, \lambda); H). \tag{4.4}$$

Of course, this difference only makes sense for $\lambda < \inf \sigma_{\mathrm{ess}}(H_0)$; if the interval $(-\infty, \lambda)$ contains points of the essential spectrum of H_0 and H, then we formally obtain $\infty - \infty$. However, there is a natural regularisation of the difference (4.4) which is well defined for all $\lambda \in \mathbb{R} \setminus \sigma_{\mathrm{ess}}(H_0)$. This regularisation is given by

$$\Xi(\lambda; H, H_0) = \mathrm{index}(E((-\infty, \lambda); H_0), E((-\infty, \lambda); H)),$$

where the r.h.s. is the Fredholm index of a pair of projections (see [1] for the definition and a survey of the index of a pair of projections). The index $\Xi(\lambda; H, H_0)$ appeared in spectral theory in various guises, mostly in connection with the spectral shift function theory, see e.g. [6, 5, 3, 7]. It was systematically surveyed and studied in [14]. The properties of $\Xi(\lambda; H, H_0)$ relevant to us are monotonicity,

$$H^- \leq H \leq H^+ \quad \Rightarrow \quad \Xi(\lambda; H^-, H_0) \leq \Xi(\lambda; H, H_0) \leq \Xi(\lambda; H^+, H_0), \tag{4.5}$$

and the connection with the eigenvalue counting function,

$$\Xi(\lambda_1; H, H_0) - \Xi(\lambda_2; H, H_0) = N([\lambda_1, \lambda_2); H) - N([\lambda_1, \lambda_2); H_0), \qquad (4.6)$$

for any interval $[\lambda_1, \lambda_2]$ which does not contain points of essential spectrum of H_0 and H. In particular, taking $\lambda_1 = -\infty$, we see that $\Xi(\lambda; H, H_0)$ coincides with the difference (4.4) whenever the latter makes sense.

Proof of Theorem 4.1(ii). Let us prove the upper bound in (4.3). Using (4.5) and (4.6), we obtain

$$N([\lambda, a); H) = \Xi(\lambda; H, H_0) - \Xi(a; H, H_0) \leq \Xi(\lambda; H_\epsilon^+, H_0) - \Xi(a; H, H_0)$$
$$= N([\lambda, a); H_\epsilon^+) + \Xi(a; H_\epsilon^+, H_0) - \Xi(a; H, H_0) = N([\lambda, a); H_\epsilon^+) + k_+,$$

for any $\lambda \in (0, a)$, where k_+ is independent of λ. It follows that

$$N((\lambda, a); H) \leq N((\lambda, a); H_\epsilon^+) + k_+, \quad \forall \lambda \in (0, a). \qquad (4.7)$$

Next,

$$N((\lambda, a); H_\epsilon^+) = N((\lambda, a); P_0 V_\epsilon^+ P_0) + N((\lambda, a); P_0^\perp (H_0 + V_\epsilon^+) P_0^\perp)$$
$$\leq N((\lambda, a); P_0 V_\epsilon^+ P_0) + N((0, a); P_0^\perp (H_0 + V_\epsilon^+) P_0^\perp) \leq N((\lambda, \infty); P_0 V_\epsilon^+ P_0) + l_+, \quad (4.8)$$

where $l_+ = N((0, a); P_0^\perp (H_0 + V_\epsilon^+) P_0^\perp)$. Combining (4.7) and (4.8), we obtain the upper bound in (4.3) with $m = l_+ + k_+$.

Next, in order to prove the lower bound in (4.3), similarly to (4.7), (4.8), we obtain

$$N((\lambda, a); H) \geq N((\lambda, a); H_\epsilon^-) - k_-, \quad \forall \lambda \in (0, a),$$

for some $k_- < \infty$, and

$$N((\lambda, a); H_\epsilon^-) = N((\lambda, a); P_0 V_\epsilon^- P_0) + N((\lambda, a); P_0^\perp (H_0 + V_\epsilon^-) P_0^\perp)$$
$$\geq N((\lambda, a); P_0 V_\epsilon^- P_0) = N((\lambda, \infty); P_0 V_\epsilon^- P_0) - N([a, \infty); P_0 V_\epsilon^- P_0).$$

Combining the last two estimates, we obtain the lower bound in (4.3) with $m = k_- + N([a, \infty); P_0 V_\epsilon^- P_0)$. ∎

4.3. **Connection between $P_0 V P_0$ and T_V.** Recall (see e.g. [4, Section 4]) that the operator $P_0 V P_0$ is unitarily equivalent to the Toeplitz operator T_V; in particular,

$$N((\lambda, \infty); \pm T_V) = N((\lambda, \infty); \pm P_0 V P_0), \quad \forall \lambda > 0. \qquad (4.9)$$

Thus, we can apply the results of the previous Sections to the study of the splitting of the lowest Landau level. Suppose that V is of the form (1.8), where the closed convex hulls of the Lipschitz domains Ω_- and Ω_+ are disjoint and the functions $v_\pm > 0$ are bounded and separated from zero. Then by choosing $\epsilon > 0$ sufficiently small, we can ensure that $V_\epsilon^\pm = V \pm \epsilon |V|$ have the form

$$V_\epsilon^\pm = v_+^{(\pm)} \chi_{\Omega_+} - v_-^{(\pm)} \chi_{\Omega_-}$$

with $v_+^{(\pm)}, v_-^{(\pm)}$ also bounded and separated from zero. Thus, Theorem 1.2 applies to $T_{V_\epsilon^\pm}$. Combining this with (4.9) and using Theorem 4.1, we arrive at the following result.

Theorem 4.3. *Under the above assumptions on the potential V, the lowest Landau level necessarily generates two infinite sequences of eigenvalues of H converging to zero. For any $a \in (0, 2)$ we have*

$$N((-\infty, -\lambda); H) + N((\lambda, a); H) = \frac{|\log \lambda|}{\log |\log \lambda|}(1 + o(1)), \quad \lambda \to +0. \qquad (4.10)$$

For any $a \in (0, 2)$ and some $c > 0$ we have

$$N((-\infty, -\lambda); H) \geq c\frac{|\log \lambda|}{\log |\log \lambda|}, \quad N((\lambda, a); H) \geq c\frac{|\log \lambda|}{\log|\log \lambda|} \qquad (4.11)$$

for all sufficiently small $\lambda > 0$.

Following the same pattern, one can also apply Theorems 2.1, 2.2 and Corollary 3.7 to the analysis of the eigenvalues of H.

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